4.7 INVERSE TRIGONOMETRIC FUNCTIONS

Copyright © Cengage Learning. All rights reserved.

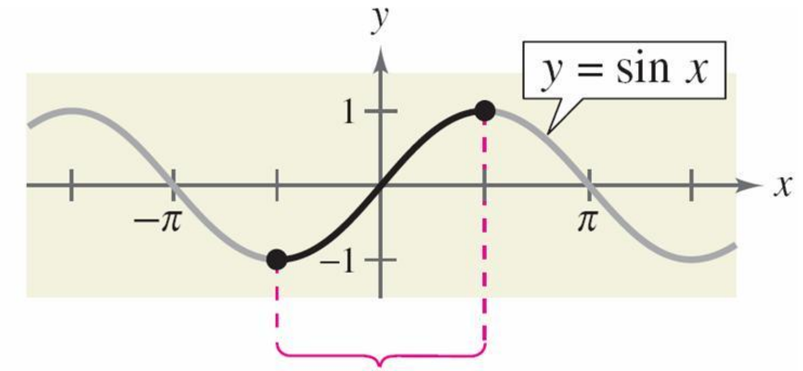
What You Should Learn

- Evaluate and graph the inverse sine function.
- Evaluate and graph the other inverse trigonometric functions.
- Evaluate and graph the compositions of trigonometric functions.

Inverse Sine Function

Inverse Sine Function

For a function to have an inverse function, it must be one-to-one—that is, it must pass the Horizontal Line Test.



sin *x* has an inverse function

on this interval.

By definition, the values of inverse trigonometric functions are *always in radians*.

Restrict the domain to the interval $-\pi/2 \le x \le \pi/2$, the following properties hold.

1. On the interval $[-\pi/2, \pi/2]$, the function $y = \sin x$ is increasing.

2. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range of values, $-1 \le \sin x \le 1$.

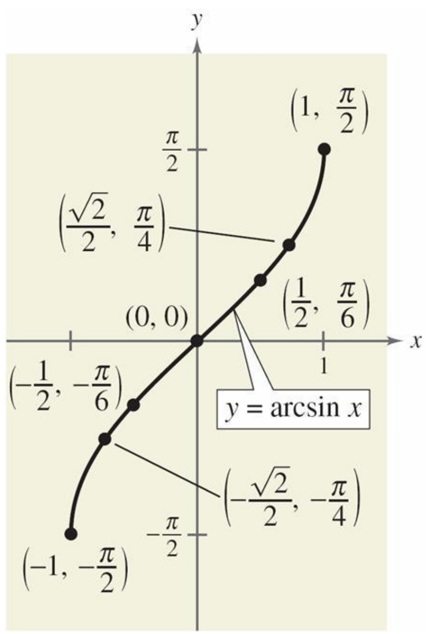
3. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ is one-to-one.

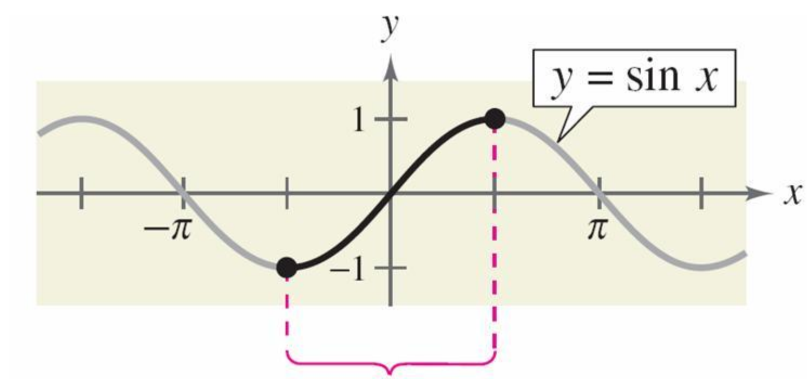
Inverse Sine Function

On the restricted domain $-\pi/2 \le x \le \pi/2$, $y = \sin x$ has a unique inverse function called the **inverse sine function.**

$$y = \arcsin x \quad \text{or} \quad y = \sin^{-1} x.$$

means the angle (or arc) whose sine is x.





Inverse Sine Function

Definition of Inverse Sine Function

The **inverse sine function** is defined by

$$y = \arcsin x \qquad \text{if and only if} \qquad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$. The domain of $y = \arcsin x$ is $[-1, 1]$, and the range is $[-\pi/2, \pi/2]$.

Example 1 – *Evaluating the Inverse Sine Function*

If possible, find the exact value.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\sin^{-1}\frac{\sqrt{3}}{2}$ **c.** $\sin^{-1} 2$

Solution:

a. Because $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$,

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$ Angle whose sine is $-\frac{1}{2}$

Example 1 – *Solution*

cont'd

b. Because $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$,

$$\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}.$$

Angle whose sine is $\sqrt{3}/2$

c. It is not possible to evaluate $y = \sin^{-1} x$ when $x = 2$ because there is no angle whose sine is 2.

Remember that the domain of the inverse sine function is [–1, 1].

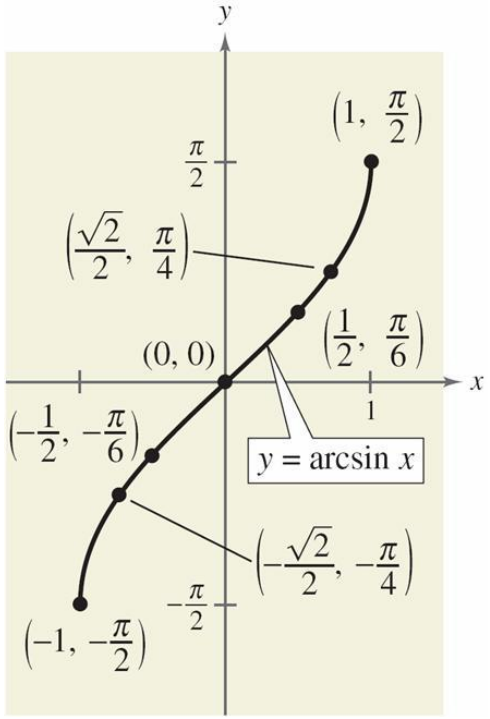
Example 2 – *Graphing the Arcsine Function*

Sketch a graph of $y = \arcsin x$.

Solution:

In the interval $[-\pi/2, \pi/2]$,

y	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x = \sin y$	-1	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1



Example 2 – *Solution*

cont'd

$y = \arcsin x$

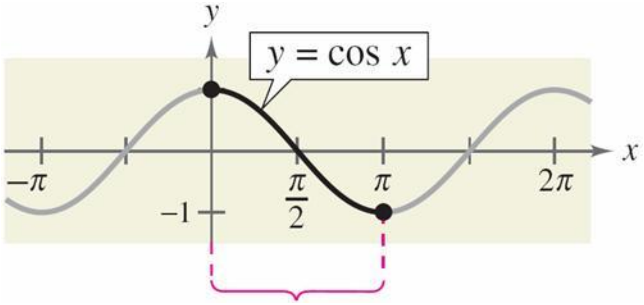
Domain: [–1, 1]

Range: $[-\pi/2, \pi/2]$

Other Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

The cosine function is decreasing and one-to-one on the interval $0 \le x \le \pi$.



cos *x* has an inverse function on this interval.

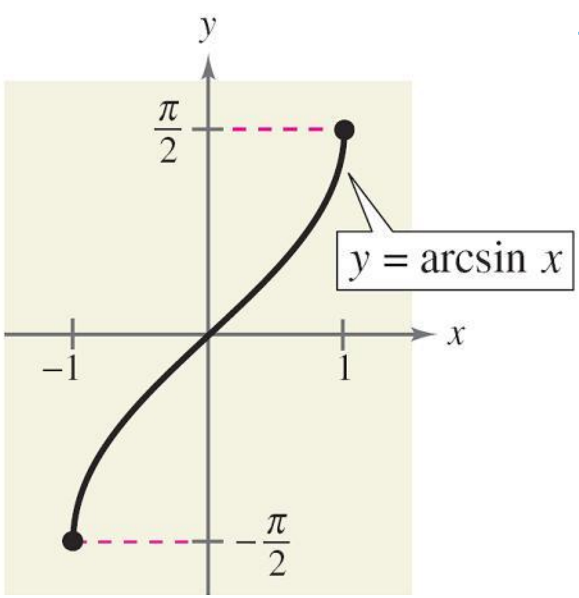
On the interval $0 \le x \le \pi$ the cosine function has an inverse function—the **inverse cosine function**:

$$y = \arccos x \quad \text{or} \quad y = \cos^{-1} x.$$

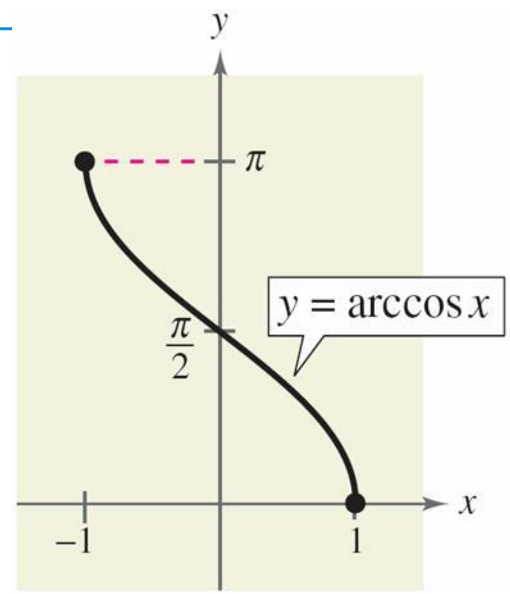
Other Inverse Trigonometric Functions

Definitions of the Inverse Trigonometric Functions

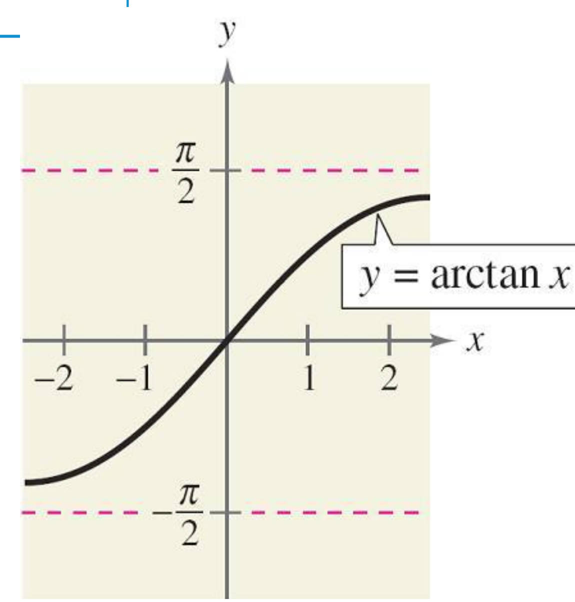
Function	*Domain*	*Range*
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$



DOMAIN: [–1,1]
RANGE: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



DOMAIN: [–1,1]
RANGE: [0, π]



DOMAIN: $(-\infty, \infty)$
RANGE: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example 3 – *Evaluating Inverse Trigonometric Functions*

Find the exact value.

a. $\arccos \dfrac{\sqrt{2}}{2}$

b. $\cos^{-1}(-1)$

c. arctan 0

d. $\tan^{-1}(-1)$

Solution:

a. Because $\cos(\pi/4) = \sqrt{2}/2$, and $\pi/4$ lies in $[0, \pi]$,

$$\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}.$$

Angle whose cosine is $\sqrt{2}/2$

Example 3 – *Solution*

cont'd

b. Because $\cos \pi = -1$, and π lies in $[0, \pi]$,

$$\cos^{-1}(-1) = \pi.$$

Angle whose cosine is –1

c. Because $\tan 0 = 0$, and 0 lies in $(-\pi/2, \pi/2)$,

$$\arctan 0 = 0.$$

Angle whose tangent is 0

Example 3 – *Solution*

cont'd

d. Because $\tan(-\pi/4) = -1$, and $-\pi/4$ lies in $(-\pi/2, \pi/2)$,

$$\tan^{-1}(-1) = -\frac{\pi}{4}.$$

Angle whose tangent is –1

Compositions of Functions

Compositions of Functions

For all x in the domains of f and f^{-1}, inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Inverse Properties of Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \qquad \text{and} \qquad \arcsin(\sin y) = y.$$

If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\arccos x) = x \qquad \text{and} \qquad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \qquad \text{and} \qquad \arctan(\tan y) = y.$$

Compositions of Functions

These inverse properties do not apply for arbitrary values of *x* and *y*.

$$\arcsin\left(\sin \frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2} \neq \frac{3\pi}{2}.$$

The property $\arcsin(\sin y) = y$

is not valid for values of *y* outside the interval $[-\pi/2, \pi/2]$.

Example 5 – *Using Inverse Properties*

If possible, find the exact value.

a. tan[arctan(–5)] **b.** $\arcsin\left(\sin \frac{5\pi}{3}\right)$ **c.** $\cos(\cos^{-1}\pi)$

Solution:

a. Because –5 lies in the domain of the arctan function, the inverse property applies, and you have

tan[arctan(–5)] = –5.

Example 5 – *Solution*

cont'd

b. In this case, $5\pi/3$ does not lie within the range of the arcsine function, $-\pi/2 \le y \le \pi/2$.

However, $5\pi/3$ is coterminal with

$$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

which does lie in the range of the arcsine function, and you have

$$\arcsin\left(\sin\frac{5\pi}{3}\right) = \arcsin\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}.$$

Example 5 – *Solution*

cont'd

c. The expression $\cos(\cos^{-1}\pi)$ is not defined because $\cos^{-1}\pi$ is not defined.

Remember that the domain of the inverse cosine function is [–1, 1].

Example 6 – *Evaluating Compositions of Functions*

Find the exact value: $\tan\left(\arccos\frac{2}{3}\right)$

Let $u = \arccos\frac{2}{3}$, then $\cos u = \frac{2}{3} > 0$, therefore, u is in QI

$$\tan\left(\arccos\frac{2}{3}\right) = \tan u = \frac{\sin u}{\cos u} = \frac{\sqrt{1-\left(\frac{2}{3}\right)^2}}{\frac{2}{3}} = \frac{\sqrt{5}}{2}$$

Example 6 – *Evaluating Compositions of Functions*

Find the exact value: $\cos\left(\arcsin(-\frac{3}{5})\right)$

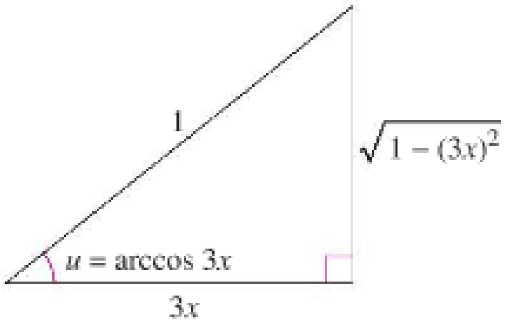
Let $u = \arcsin(-\frac{3}{5})$, then $\sin u = -\frac{3}{5} < 0$, therefore, u is in Q IV

$$\cos\left(\arcsin(-\frac{3}{5})\right) = \sqrt{1-\left(-\frac{3}{5}\right)^2} = \frac{4}{5}$$

Example

Write the expression as an algebraic expression in x:

$$\sin(\arccos 3x), 0 \leq x \leq \frac{1}{3}$$



Angle whose cosine is $3x$

$$\sin(\arccos 3x) = \sqrt{1 - 9x^2}$$